Nonstandard Methods on Representations of the Canonical Commutation Relations

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数理解析研究所講究録
1100 巻 1999 年 157-167

NONSTANDARD METHODS ON REPRESENTATIONS OF THE CANONICAL COMMUTATION RELATIONS

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1 NONSTANDARD ANALYSIS

There are several different formulations of nonstandard analysis. This paper adopts the set-theoretical approach based on superstructures instituted by Robinson and Zakon [2] and follows the up-to-date description by Chang and Keisler [3].

For any set X, let $S(X)$ denote the set of all subsets of X. The *superstructure* over X, denoted by $V(X)$, is defined by the following recursion:

$$V_0(X) = X, \quad V_{n+1}(X) = V_n(X) \cup S(V_n(X)), \quad V(X) = \bigcup_{n \in \mathbf{N}} V_n(X),$$

where $\mathbf{N}$ is the set of natural numbers. The set X is called a *base set* if $\emptyset \notin X$ and for all $x \in X$ we have $x \cap V(X) = \emptyset$.

The language $\mathcal{L}$ which describes $V(X)$ consists of logical connectives $\neg$, $\wedge$, $\vee$, $\Rightarrow$, quantifiers $\forall$, $\exists$, individual variables x',x'', $\ldots$, individual constants C_u for all $u \in V(X)$, and two binary predicate constants $=$, $\in$. A *formula* of $\mathcal{L}$ is constructed from the above constituents in the usual way. We will use the following abbreviations, called *bounded quantifiers*: $(\forall x \in y)\phi$ means $(\forall x)[x \in y \Rightarrow \phi]$, $(\exists x \in y)\phi$ means $(\exists x)[x \in y \wedge \phi]$. A *bounded formula* is a formula in which every quantifier occurs as a bounded quantifier. We will write $\phi[u_1, ..., u_n]$ for $\phi(C_{u_1}, ..., C_{u_n})$.

For any formula ϕ in $\mathcal{L}$, the relation $V(X) \models \phi$ is defined by the following rules:

(i) $V(X) \models C_u = C_v$ if and only if u and v are identical.

(ii) $V(X) \models C_u \in C_v$ if and only if u is an element of v.

(iii) $V(X) \models \neg\phi$ if and only if $V(X) \models \phi$ does not hold.

(iv) $V(X) \models \phi_1 \wedge \phi_2$ if and only if $V(X) \models \phi_1$ and $V(X) \models \phi_2$.

(v) $V(X) \models \phi_1 \vee \phi_2$ if and only if $V(X) \models \phi_1$ or $V(X) \models \phi_2$.

(vi) $V(X) \models \phi_1 \Rightarrow \phi_2$ if and only if $V(X) \models \phi_1$ then $V(X) \models \phi_2$.

(vii) $V(X) \models (\forall x)\phi(x)$ if and only if $V(X) \models \phi[u]$ for all u in $V(X)$.

(viii) $V(X) \models (\exists x)\phi(x)$ if and only if $V(X) \models \phi[u]$ for some u in $V(X)$.

A *nonstandard universe* is a triple $\langle V(X), V(Y), \star\rangle$ consisting of superstructures $V(X)$, $V(Y)$, and a map $\star : V(X) \to V(Y)$ satisfying the following conditions (i)–(iii):

(i) X and Y are infinite base sets.

(ii) **(Transfer Principle)** The map $\star : a \mapsto {}^\star a$ is an injective mapping from $V(X)$ into $V(Y)$, and for any bounded formula $\phi(x_1, \ldots, x_n)$ in $\mathcal{L}$,

$$V(X) \models \phi[u_1, \ldots, u_n] \quad \text{if and only if} \quad V(Y) \models \phi[{}^\star u_1, \ldots, {}^\star u_n]$$

for any $u_1, \ldots, u_n$ in $V(X)$.

(iii) ${}^\star X = Y$.

An element $u \in V(Y) \setminus Y$ is called an *internal set* if there is $x \in V(X)$ such that $u \in {}^\star x$. Let α be a cardinal. A nonstandard universe $\langle V(X), V(Y), \star\rangle$ is said to be *α-saturated* if it satisfies the following condition:

(iv) **(Saturation Principle)** Every family of less than α internal sets with the finite intersection property has nonempty intersection.

In this paper, we always work with a nonstandard universe $\langle V(X), V(Y), \star\rangle$ which is α-saturated with $\operatorname{card}(V(X)) < \alpha$; such a nonstandard universe is said to be *polysaturated.* We also assume that the base X includes the complex numbers $\mathbf{C}$ and any other structures under consideration such as given groups and Hilbert spaces.

For a set S, let ${}^\sigma S = \{{}^\star s |\ s \in S\}$. We identify ${}^\star z$ with z for all $z \in \mathbf{C}$. Hence, ${}^\sigma S = S$ if S is a subset of $\mathbf{C}$, e.g., ${}^\sigma\mathbf{C} = \mathbf{C}$, ${}^\sigma\mathbf{R} = \mathbf{R}$ (the real numbers), ${}^\sigma\mathbf{Z} = \mathbf{Z}$ (the integers), and ${}^\sigma\mathbf{N} = \mathbf{N}$. Let $\mathbf{R}^+$, ${}^\star\mathbf{R}_0$, ${}^\star\mathbf{R}_0^+$, ${}^\star\mathbf{R}_\infty^+$, and ${}^\star\mathbf{N}_\infty$ denote the sets of positive real numbers, infinitesimal hyperreal numbers, positive infinitesimal hyperreal numbers, positive infinite hyperreal numbers and infinite hypernatural numbers, respectively. It is shown that ${}^*\mathbf{N}_\infty = {}^*\mathbf{N}\backslash\mathbf{N}$. We write $x \sim \infty$ if $x \in \mathbf{R}^+_\infty$, and $0 < x < \infty$ if $x \in \mathrm{fin}{}^*\mathbf{R}^+ = {}^*\mathbf{R}^+\backslash{}^*\mathbf{R}^+_\infty$. If $r \in {}^*\mathbf{R}$ and $|r| < \infty$, the standard part of r is denoted by ${}^\circ r$. If $r \sim \infty$, we write ${}^\circ r = \infty$. Let $x, y \in {}^*\mathbf{R}^+$. we say that x is of the *order* of y, in symbols $x \asymp y$, iff $0 < x/y < \infty$ and $0 < y/x < \infty$. We write $x \ll y$ if $x/y \approx 0$. For a hyperfinite ($\star$-finite) set F, let $|F|$ denote the internal cardinal number of F.

Let $(X, \mathcal{O})$ be a topological space. Let $\mathcal{O}_x$ denote the system of open neighborhoods of $x \in X$. The *monad* of $x \in X$ is the subset of ${}^*X$ defined by $\mathrm{mon}_\mathcal{O}(x) = \bigcap\{{}^*O|\ O \in \mathcal{O}_x\}$. The set of *near standard* points is the subset of ${}^*X$ defined by $\mathrm{ns}({}^*X) = \bigcup\{\mathrm{mon}_\mathcal{O}(x)|\ x \in X\}$. It is shown that $(X, \mathcal{O})$ is Hausdorff if and only if $x \neq y$ implies $\mathrm{mon}_\mathcal{O}(x) \cap \mathrm{mon}_\mathcal{O}(y) = \emptyset$. Thus for any Hausdorff space $(X, \mathcal{O})$, we can define the equivalence relation $\approx_\mathcal{O}$ on $\mathrm{ns}{}^*X$ so that $a \approx_\mathcal{O} b$ iff $a \in \mathrm{mon}_\mathcal{O}(x)$ and $b \in \mathrm{mon}_\mathcal{O}(x)$ for some $x \in X$. Let $(X, \|\cdot\|)$ be an internal normed linear space. Define the relation $\approx$ on X so that $x \approx y$ iff $\|x - y\| \approx 0$. The *principal galaxy* of X is the subset of ${}^*X$ defined by $\mathrm{fin}(X) = \{x \in X|\ \|x\| < \infty\}$. For $x \in \mathrm{fin}(X)$, let $\hat{x}$ denote the equivalence class $\hat{x} = \{y \in X|\ x \approx y\}$. Let $\hat{X} = \{\hat{x}|\ x \in \mathrm{fin}(X)\}$. Define the (standard) norm $\|\cdot\|$ on $\hat{X}$ by $\|\hat{x}\| = {}^\circ\|x\|$ for all $x \in \mathrm{fin}X$. Then $(\hat{X}, \|\cdot\|)$ turns out to be a Banach space, called the *standardization* of $(X, \|\cdot\|)$. In a similar way, the standardization is defined for any internal pre-Hilbert space $(X, \langle\cdot,\cdot\rangle)$, and it turns to be a Hilbert space.

For a (standard) normed linear space $(X, \|\cdot\|)$, we abbreviate $\widehat{{}^*X}$ to $\hat{X}$. In this case, the Banach space $(\hat{X}, \|\cdot\|)$ is called the *nonstandard hull* of $(X, \|\cdot\|)$.

Let $\mathcal{H}$ be an internal Hilbert space, and $T : \mathcal{H} \to \mathcal{H}$ an internal bounded operator such that the bound $\|T\|$ is finite. The bounded operator $\hat{T} : \hat{\mathcal{H}} \to \hat{\mathcal{H}}$, called the *standardization* of T, is defined by the relation $\hat{T}\hat{x} = \widehat{Tx}$.

For further information on nonstandard real analysis, we refer to Stroyan and Luxemburg [5] and Hurd and Loeb [4].

2 CANONICAL COMMUTATION RELATIONS

Let $\mathcal{H}$ be a Hilbert space, and $\{a_i|i \in I\}$ a family of linear operators on $\mathcal{H}$, with dense domains $\mathrm{dom}(a_i)$, when I is finite or infinite. Let D be a dense subspace of $\mathcal{H}$. The pair $(\{a_i\}, D)$ is called a *representation of the CCR* if the following conditions are satisfied for all i:

(i) $D \subset \mathrm{dom}(a_i)$, $D \subset \mathrm{dom}(a_i^*)$.

(ii) D is invariant under a_i and a_i^*.

(iii) $[a_i, a_j^*] = \delta_{ij}$, $[a_i, a_j] = 0$ on D.

The number $n = |I|$ is called the *degree of freedom* of the CCR.

Let $U(\cdot)$ and $V(\cdot)$ be strongly continuous one-parameter unitary groups on $\mathcal{H}$. The pair (U, V) is a *representation of the Weyl CCR* (of one degree of freedom) if

$$U(p)V(q) = e^{ipq}V(q)U(p) \tag{1}$$

for all $p, q \in \mathbf{R}$.

Let $\mathbf{H}_1$ be the group $(\mathbf{R} \times \mathbf{R} \times \mathbf{R}, \cdot)$ with group law

$$(p, q, t) \cdot (p', q', t') = (p + p', q + q', t + t' + pq').$$

Then, we get a strongly-continuous representation of $\mathbf{H}_1$ by

$$(p, q, t) \mapsto W(p, q, t) := e^{it}V(q)U(p),$$

where (U, V) is a representation of the Weyl CCR. The representation W is called a *Weyl representation* of $\mathbf{H}_1$.

Let $U(p)$ and $V(q)$ be the unitary operators on $L^2(\mathbf{R})$ defined by

$$(U(p)f)(x) = e^{ip}f(x),$$

$$(V(q)f)(x) = f(x - t).$$

Then, $\rho(p, q, t) := e^{it}V(q)U(p)$ is a Weyl representation of $\mathbf{H}_1$. Let us call this the *Schrödinger representation* of $\mathbf{H}_1$.

3 NONSTANDARD REPRESENTATIONS OF THE CCR

Let $\nu \in {}^*\mathbf{N}$ be infinite, and $\{a_i | i \in \mathbf{N}\}$ be a sequence of internal operators on ${}^*\mathbf{C}^\nu$, or equivalently, be $\nu \times \nu$ internal matrices. Let $D \subset \mathrm{fin}{}^*\mathbf{C}^\nu$ be an external subspace, invariant under a_i and a_i^* for all i. The pair $(\{a_i\}, D)$ is called a *hyperfinite representation of the CCR* if

$$\|[a_i, a_j^*]\xi - \delta_{ij}\xi\| \approx 0,$$

for all $\xi \in D$.

We easily see the following:

Lemma 3.1 *If $\xi, \eta \in D$ and $\xi \approx \eta$, then $a_i\xi \approx a_i\eta$ and $a_i^*\xi \approx a_i^*\eta$.*

This allows us to define the operators $\hat{a}_i$ and $\hat{a}_i^*$ on the Hilbert space $\hat{D}^{\perp\perp}$ by

$$\hat{a}_i\hat{\xi} = \widehat{a_i\xi}, \qquad \hat{a}_i^*\hat{\xi} = \widehat{a_i^*\xi}.$$

We call $\hat{a}_i$ and $\hat{a}_i^*$ the *standard part* of a_i and a_i^*, respectively.

4 HYPERFINITE HEISENBERG GROUP

This section reviews the results given by Ojima and Ozawa [7].

Let $K \in {}^*\mathbf{N}$ be infinite, and $\mathbf{K} = \langle {}^*\mathbf{Z}/K{}^*\mathbf{Z}, \otimes, \oplus \rangle$ be a ring of residue classes modulo K. Define an inner product on ${}^*\mathbf{C}^{\mathbf{K}}$ by

$$\langle f, g \rangle := \sum_{k \in \mathbf{K}} \overline{f(k)} g(k) \Delta x,$$

for $f, g \in {}^*\mathbf{C}^{\mathbf{K}}, \Delta x \in {}^*\mathbf{R}, \Delta x > 0$. Define $\mathbf{H}$ to be $\mathbf{K} \times \mathbf{K} \times \mathbf{K}$ equipped with the group law

$$(k, l, m)(k', l', m') = (k \oplus k', l \oplus l', m \oplus m' \oplus (k \otimes l')).$$

Let us call $\mathbf{H}$ the *hyperfinite Heisenberg group*. Let $W(k, l, m)$ be the internal operator on ${}^*\mathbf{C}^{\mathbf{K}}$ defined by $W(k, l, m)f(k') = e^{2\pi i(m + lk')/K} f(k' \oplus k)$.

Proposition 4.1 *The map $W : (k, l, m) \mapsto W(k, l, m)$ is an internal irreducible unitary representation of $\mathbf{H}$.*

We call it the *hyperfinite Schrödinger representation* of $\mathbf{H}$.

Proposition 4.2 *The map $\hat{W}$ is a unitary representation of $\mathbf{H}$ on $\widehat{{}^*\mathbf{C}^{\mathbf{K}}}$.*

Let $\Delta x = K^{-1/2}$, and $\Delta(p, q, t) = (\lfloor p/\Delta x \rfloor, \lfloor q/\Delta x \rfloor, \lfloor t/\Delta x \rfloor)$.

Theorem 4.3 *Let* $\mathrm{fin}(\mathbf{H})$ *be the subgroup of* $\mathbf{H}$ *defined by*

$$\mathrm{fin}(\mathbf{H}) = \{(k,l,m) \mid |k\Delta x| < \infty, |l\Delta x| < \infty, |m\Delta x^2| < \infty\}.$$

Then, there is $f \in \mathrm{fin}({}^*\mathbf{C}^{\mathbf{K}})$ *satisfying the following. Let* $\mathcal{H}$ *be the closed subspace of* $\widehat{{}^*\mathbf{C}^{\mathbf{K}}}$ *such that*

$$\mathcal{H} = \{\hat{W}(k,l,m)\hat{f} \mid (k,l,m) \in \mathrm{fin}(\mathbf{H})\}^{\perp\perp}.$$

For any $(k,l,m) \in \mathrm{fin}(\mathbf{H})$, *let* $\tilde{W}(k,l,m)$ *be the restriction of* $\hat{W}(k,l,m)$ *to* $\mathcal{H}$. *Then the map* $(p,q,t) \mapsto \tilde{W}(\Delta(p,q,t))$ *is a strongly continuous unitary representation of* $\mathbf{H}_1$, *unitarily equivalent to the Schrödinger representation* ρ.

5 HYPERFINITE PARAFERMI OPERATORS

This section reviews the results given by Yamashita [7].

Let $\nu \in \mathbf{N}$ and $d \in \mathbf{N}$. Suppose that $b_1, ..., b_\nu \in M(d, \mathbf{C})$ (i.e., $b_1, ..., b_\nu$ are finite-dimensional matrices). The matrices $b_1, ..., b_\nu$ are called the *annihilation operators* of parafermi oscillators of order $p \in \mathbf{N}$ if they satisfy

$$[b_k, [b_l{}^*, b_m]] = 2\delta_{k\,l} b_m,$$
$$[b_k, [b_l{}^*, b_m{}^*]] = 2\delta_{k\,l} b_m{}^* - 2\delta_{k\,m} b_l,{}^*$$
$$[b_k, [b_l, b_m]] = 0,$$

and the uniqueness of vacuum $|0\rangle$, and,

$$b_k b_l{}^* |0\rangle = \delta_{k\,l} p |0\rangle.$$

The matrices $b_1^*, ..., b_\nu^*$ are called the *creation operators* of parafermi oscillators of order p. The *hyperfinite annihilation operators* of parafermi oscillators are the internal matrices defined by substituting ${}^*\mathbf{N}$ and ${}^*\mathbf{C}$ for $\mathbf{N}$ and $\mathbf{C}$ in the above definition, respectively.

Green [8] has given a class of representations of the above commutation relations of the parafermi creation and annihilation operators. In the so-called the *Green representation* for the cases of order p, the parafermi operators b_k are expressed by the form

$$b_k = \sum_{\alpha=1}^{p} b_k^{(\alpha)},$$

where the *Green-component* operators $b_k^{(\alpha)}$ satisfy the commutation relations

$$\{b_k^{(\alpha)}, b_l^{(\alpha)^*}\} = \delta_{kl}, \quad \{b_k^{(\alpha)}, b_l^{(\alpha)}\} = 0,$$

$$[b_k^{(\alpha)}, b_l^{(\beta)^*}] = [b_k^{(\alpha)}, b_l^{(\beta)}] = 0 \quad (\alpha \neq \beta);$$

where $\{A, B\} = AB + BA$, and the uniqueness of vacuum $|0\rangle$ such that

$$b_k^{(\alpha)}|0\rangle = 0 \quad \text{for all } k, \alpha.$$

The Green representation is essentially equivalent to the tensor product representation of the Clifford algebra representation of $so(2\nu)$. In fact, we easily verify that $e_1, ..., e_{2\nu}$ defined by $e_{2k-1} = i(b_k^* + b_k)$ and $e_{2k} = b_k^* - b_k$ form the generators of a Clifford algebra, i.e., $e_i^2 = -1$ and $e_i e_j = -e_j e_i$ $(i \neq j)$. Thus, we can construct a $2^{p\nu}$-dimensional representation of Green components by using a spin representation of the Clifford algebra as follows. Let $V_k^{(\alpha)} \simeq \mathbf{C}^2$ $(k = 1, ..., \nu,\ \alpha = 1, ..., p)$. The Pauli matrices $\sigma_{1,k}^{(\alpha)}$, $\sigma_{2,k}^{(\alpha)}$ and $\sigma_{3,k}^{(\alpha)}$ that act on $V_k^{(\alpha)}$ are represented as

$$\sigma_{1,k}^{(\alpha)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_{2,k}^{(\alpha)} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_{3,k}^{(\alpha)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Define $V^{(\alpha)}$ by

$$V^{(\alpha)} = V_1^{(\alpha)} \otimes \cdots \otimes V_\nu^{(\alpha)},$$

and define $\hat{\sigma}_{c,k}^{(\alpha)}$ and $\gamma_i^{(\alpha)}$ $(k = 1, ..., \nu,\ i = 1, ..., 2\nu)$ that acts on $V^{(\alpha)}$ by

$$\hat{\sigma}_{c,k}^{(\alpha)} = \overbrace{1 \otimes \cdots \otimes 1}^{k-1} \otimes \sigma_{c,k}^{(\alpha)} \otimes 1 \otimes \cdots \otimes 1, \quad c = 1, 2, 3,$$

$$\gamma_{2k-1}^{(\alpha)} = \hat{\sigma}_{2,k}^{(\alpha)} \hat{\sigma}_{3,k+1}^{(\alpha)} \cdots \hat{\sigma}_{3,\nu}^{(\alpha)},$$

$$\gamma_{2k}^{(\alpha)} = -\hat{\sigma}_{1,k}^{(\alpha)} \hat{\sigma}_{3,k+1}^{(\alpha)} \cdots \hat{\sigma}_{3,\nu}^{(\alpha)}.$$

The operators $b_k^{(\alpha)}$ $(k = 1, ..., \nu)$ defined by

$$b_k^{(\alpha)} = \frac{1}{2}(\gamma_{2k-1}^{(\alpha)} - i\gamma_{2k}^{(\alpha)}),$$

satisfy the relations

$$\{b_k^{(\alpha)}, b_l^{(\alpha)*}\} = \delta_{kl}, \quad \{b_k^{(\alpha)}, b_l^{(\alpha)}\} = 0,$$

for all $k,\ l = 1, ..., \nu$.

Define V and $\tilde{b}_k^{(\alpha)}$ $(k = 1, ..., \nu)$ acting on V by

$$V = V^{(1)} \otimes \cdots \otimes V^{(p)},$$

$$\tilde{b}_k^{(\alpha)} = \overbrace{1 \otimes \cdots \otimes 1}^{\alpha-1} \otimes b_k^{(\alpha)} \otimes 1 \otimes \cdots \otimes 1.$$

We see that for all $k, l = 1, ..., \nu$,

$$\{\tilde{b}_k^{(\alpha)}, \tilde{b}_l^{(\alpha)*}\} = \delta_{k\,l}, \quad \{\tilde{b}_k^{(\alpha)}, \tilde{b}_l^{(\alpha)}\} = 0,$$

$$[\tilde{b}_k^{(\alpha)}, \tilde{b}_l^{(\beta)*}] = [\tilde{b}_k^{(\alpha)}, \tilde{b}_l^{(\beta)}] = 0, \quad (\alpha \neq \beta).$$

Let $|0\rangle_k^{(\alpha)} \in V_k^{(\alpha)}$ denote the unit vector satisfying $b_k^{(\alpha)}|0\rangle_k^{(\alpha)} = 0$. Define $|0\rangle^{(\alpha)}$ and $|0\rangle$ by

$$|0\rangle^{(\alpha)} = |0\rangle_1^{(\alpha)} \otimes \cdots \otimes |0\rangle_\nu^{(\alpha)},$$

$$|0\rangle = |0\rangle^{(\alpha)} \otimes \cdots \otimes |0\rangle^{(p)}.$$

Now, we find that $\tilde{b}_1^{(\alpha)}, ..., \tilde{b}_\nu^{(\alpha)}$ are $2^{p\nu}$-dimensional representations of the Green components and $|0\rangle$ is the vacuum. Thus, $b_k = \sum_{\alpha=1}^{p} \tilde{b}_k^{(\alpha)}$, $(k = 1, ..., \nu)$ are $2^{p\nu}$-dimensional representations of annihilation operators of ν parafermi oscillators of order p. Let us call the above representation of the algebra of the parafermi oscillators the *spin representation.*

Define $\sigma_{\pm,k}^{(\alpha)}$ by

$$\sigma_{\pm,k}^{(\alpha)} = (\sigma_{1,k}^{(\alpha)} \pm i\sigma_{2,k}^{(\alpha)})/2,$$

and $|1\rangle_k^{(\alpha)} \in V_k^{(\alpha)}$ by

$$|1\rangle_k^{(\alpha)} = \sigma_{+,k}^{(\alpha)}|0\rangle_k^{(\alpha)}.$$

The set of vectors

$$\{(|e_1^{(1)}\rangle_1^{(1)} \cdots |e_\nu^{(1)}\rangle_\nu^{(1)}) \cdots (|e_1^{(p)}\rangle_1^{(p)} \cdots |e_\nu^{(p)}\rangle_\nu^{(p)}) : e_k^{(\alpha)} = 0, 1\}$$

($\otimes$'s are omitted) is a complete orthonormal system of V. We write the vectors simply as $|\{e_k^{(\alpha)}\}\rangle$.

The number operator N on V and the related operators $N_k, N^{(\alpha)}$ are defined as follows:

$$N_k^{(\alpha)} = \frac{1}{2}(1 + \sigma_{3,k}^{(\alpha)}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\begin{aligned}
\hat{N}_k^{(\alpha)} &= \overbrace{1 \otimes \cdots \otimes 1}^{k-1} \otimes N_k^{(\alpha)} \otimes \overbrace{1 \otimes \cdots \otimes 1}^{\nu-k}, \\
\tilde{N}_k^{(\alpha)} &= \overbrace{1 \otimes \cdots \otimes 1}^{\alpha-1} \otimes \hat{N}_k^{(\alpha)} \otimes \overbrace{1 \otimes \cdots \otimes 1}^{p-\alpha}, \\
N_k &= \sum_{\alpha=1}^{p} \tilde{N}_k^{(\alpha)}, \;\; N^{(\alpha)} = \sum_{k=1}^{\nu} \tilde{N}_k^{(\alpha)}, \;\; N = \sum_{\alpha=1}^{p} N^{(\alpha)},
\end{aligned}$$

We see that

$$N|\{e_k^{(\alpha)}\}\rangle = n|\{e_k^{(\alpha)}\}\rangle,$$

where n is the number of $e_k^{(\alpha)}$'s that is equal to 1. It is easily shown that

$$\tilde{b}_k^{(\alpha)*}\tilde{b}_k^{(\alpha)} = \tilde{N}_k^{(\alpha)}, \;\; \tilde{b}_k^{(\alpha)}\tilde{b}_k^{(\alpha)*} = 1 - \tilde{N}_k^{(\alpha)}, \;\; N_k = \frac{1}{2}([b_k^*, b_k] + p),$$

$$[N_k, N_l] = 0, \quad N_k b_k = b_k(N_k - 1), \;\; N_k b_k^* = b_k^*(N_k + 1), \;\; \text{etc.}$$

Lemma 5.1 *Suppose that the hyperfinite parafermi annihilation operators $b_1, ..., b_\nu$ are represented by the spin representation, and that their order p is an infinite hypernatural number ($b_1, ..., b_\nu$ are $2^{p\nu} \times 2^{p\nu}$ internal matrices acting on ${}^*\mathbf{C}^{2^{p\nu}}$). If $|\xi\rangle \in {}^*\mathbf{C}^{2^{p\nu}}$ satisfies $\langle\xi|\xi\rangle$, $\langle\xi|N^2|\xi\rangle < \infty$, and $k \neq l$ $(k, l = 1, 2, ..., \nu)$, then*

(i) $[\beta_k, \beta_l]|\xi\rangle \approx [\beta_k, \beta_l{}^*]|\xi\rangle \approx 0,$

(ii) $[\beta_k, \beta_k{}^*]|\xi\rangle \approx |\xi\rangle,$

(iii) $\beta_k\beta_k^{*n}|\xi\rangle \approx (\beta_k^{*n}\beta_k + n\beta_k^{*n-1})|\xi\rangle,$

where $\beta_k = p^{-1/2}b_k$ (the normalization of b_k) and $n < \infty$.

Suppose that the number of the parafermi oscillators ν and their order p are infinite hypernatural numbers. When n_i is a nonnegative integer for any $i = 1, 2, ... < \infty$, and the number of n_i's such that $n_i \neq 0$ is finite, we will define $|n_1, n_2, ...\rangle$ by

$$|n_1, n_2, ...\rangle = \frac{b_1^{*n_1} b_2^{*n_2} \cdots |0\rangle}{||b_1^{*n_1} b_2^{*n_2} \cdots |0\rangle||}.$$

Since $b_1^{*n_1} b_2^{*n_2} \cdots$ is the product of a finite number of operators, it is well-defined. $N_k|n_1, n_2, ...\rangle = n_k|n_1, n_2, ...\rangle$ is easily shown, and hence, since N_k is hermitian, the set of the vectors of the form $|n_1, n_2, ...\rangle$ is an orthonormal system.

Lemma 5.2 *The following relations hold:*

(i) $\beta_k^*\beta_k|n_1, n_2, ...\rangle \approx n_k|n_1, n_2, ...\rangle$,

(ii) $\beta_k\beta_k^*|n_1, n_2, ...\rangle \approx (n_k + 1)|n_1, n_2, ...\rangle$,

(iii) $\|\beta_1^{*n_1}\beta_2^{*n_2}\cdots|0\rangle\| \approx \sqrt{n_1!n_2!\cdots}$,

(iv) $\beta_k^*|n_1, n_2, ...\rangle \approx \sqrt{n_k+1}|n_1, n_2, ..., n_k + 1, ...\rangle$,

(v) $\beta_k|n_1, n_2, ...\rangle \approx \sqrt{n_k}|n_1, n_2, ..., n_k - 1, ...\rangle$.

Define a set $D \subset {}^*\mathbf{C}^{2^{p\nu}}$ by

$$D = \Big\{\frac{\beta_{k_1}^*\cdots\beta_{k_n}^*|0\rangle}{\|\beta_{k_1}^*\cdots\beta_{k_n}^*|0\rangle\|}\Big| n, k_1, ..., k_n \in \mathbf{N}\Big\} \cup \{|0\rangle\}.$$

Clearly, every vector in D is a normalized eigenvector of the number operator N with a finite eigenvalue. Let S denote the external subspace of ${}^*\mathbf{C}^{2^{p\nu}}$ spanned by D, i.e.,

$$S = \{\sum_{i=1}^{n} c_i|\xi\rangle \mid c_i \in {}^*\mathbf{C}, |c_i| < \infty, n \in \mathbf{N}, |\xi\rangle \in D\}.$$

The following theorem follows from Lemma 5.1 and 5.2.

Theorem 5.3 *The pair (β_k, S) $(k \in \mathbf{N})$ is a hyperfinite representation of CCR of countably-infinite degree of freedom, i.e. S is invariant with respect to β_k and β_k^* for every $k \in \mathbf{N}$, and*

$$[\beta_k, \beta_l]|\xi\rangle \approx 0,$$

$$[\beta_k, \beta_l^*]|\xi\rangle \approx \delta_{kl}|\xi\rangle,$$

for any $|\xi\rangle \in S$. Moreover, the uniqueness of vacuum is satisfied in the following sense: if $|\xi\rangle \in S$, $\langle\xi|\xi\rangle = 1$ and $\beta_k|\xi\rangle \approx 0$ for all $k \in \mathbf{N}$, then $|\xi\rangle \approx |0\rangle$.

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